

Exercise 7

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_0^\pi \sin^{2n} \theta \, d\theta \quad (n = 1, 2, \dots).$$

$$\text{Ans. } \frac{(2n)!}{2^{2n}(n!)^2} \pi.$$

Solution

Make the substitution,

$$\begin{aligned} \alpha = 2\theta & \quad \rightarrow \quad \frac{\alpha}{2} = \theta \\ d\alpha = 2 \, d\theta & \quad \rightarrow \quad \frac{d\alpha}{2} = d\theta, \end{aligned}$$

so that the limits of integration go from 0 to 2π .

$$\begin{aligned} & \int_0^{2\pi} \sin^{2n} \frac{\alpha}{2} \left(\frac{d\alpha}{2} \right) \\ & \frac{1}{2} \int_0^{2\pi} \left(\sin \frac{\alpha}{2} \right)^{2n} d\alpha \\ & \frac{1}{2} \int_0^{2\pi} \left(\sqrt{\frac{1 - \cos \alpha}{2}} \right)^{2n} d\alpha \\ & \frac{1}{2} \int_0^{2\pi} \left(\frac{1 - \cos \alpha}{2} \right)^n d\alpha \\ & \frac{1}{2} \int_0^{2\pi} \left(\frac{1 - \frac{e^{i\alpha} + e^{-i\alpha}}{2}}{2} \right)^n d\alpha \\ & \frac{1}{2} \int_0^{2\pi} \left(\frac{2 - e^{i\alpha} - e^{-i\alpha}}{2^2} \right)^n d\alpha \\ & \frac{1}{2^{2n+1}} \int_0^{2\pi} \left(2 - e^{i\alpha} - \frac{1}{e^{i\alpha}} \right)^n d\alpha \\ & \frac{1}{2^{2n+1}} \int_0^{2\pi} \left[\frac{2e^{i\alpha} - (e^{i\alpha})^2 - 1}{e^{i\alpha}} \right]^n d\alpha \end{aligned}$$

Make the following substitution.

$$\begin{aligned} z &= e^{i\alpha} \\ dz &= ie^{i\alpha} d\alpha = iz \, d\alpha \quad \rightarrow \quad \frac{dz}{iz} = d\alpha \end{aligned}$$

Doing so changes this integral into a counterclockwise contour integral over the unit circle in the complex plane.

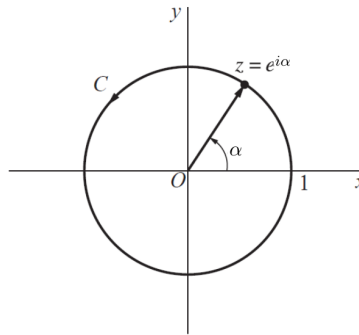


Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

$$\frac{1}{2^{2n+1}} \oint_C \left(\frac{2z - z^2 - 1}{z} \right)^n \left(\frac{dz}{iz} \right)$$

$$\frac{1}{i \cdot 2^{2n+1}} \oint_C \frac{(2z - z^2 - 1)^n}{z^{n+1}} dz$$

According to the Cauchy residue theorem, this closed contour integral is equal to $2\pi i$ times the sum of the residues at the enclosed singularities of the integrand. There's only one singular point to consider, namely $z = 0$.

$$\frac{1}{i \cdot 2^{2n+1}} \cdot 2\pi i \operatorname{Res}_{z=0} \frac{(2z - z^2 - 1)^n}{z^{n+1}}$$

$$\frac{\pi}{2^{2n}} \operatorname{Res}_{z=0} \frac{(2z - z^2 - 1)^n}{z^{n+1}}$$

Since $\phi(z) = (2z - z^2 - 1)^n$ is analytic and nonzero at 0, $z = 0$ is a pole of order $n + 1$, so the residue there can be calculated by

$$\frac{\pi}{2^{2n}} \frac{\phi^{[(n+1)-1]}(0)}{[(n+1) - 1]!}$$

$$\frac{\pi}{2^{2n}} \frac{\phi^{(n)}(0)}{n!}.$$

The goal now is to determine the n th derivative of $\phi(z)$.

$$n = 1: \quad \phi^{(1)}(z) = 2(1 - z) = 2 \cdot 1(1 - z)$$

$$n = 2: \quad \phi^{(2)}(z) = 12(1 - z)^2 = 4 \cdot 3(1 - z)^2 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1}(1 - z)^2$$

$$n = 3: \quad \phi^{(3)}(z) = 120(1 - z)^3 = 6 \cdot 5 \cdot 4(1 - z)^3 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}(1 - z)^3$$

$$n = 4: \quad \phi^{(4)}(z) = 1680(1 - z)^4 = 8 \cdot 7 \cdot 6 \cdot 5(1 - z)^4 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}(1 - z)^4$$

$$n = 5: \quad \phi^{(5)}(z) = 30240(1 - z)^5 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6(1 - z)^5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}(1 - z)^5$$

⋮

$$\phi^{(n)}(z) = \frac{(2n)!}{n!}(1 - z)^n$$

So then

$$\phi^{(n)}(0) = \frac{(2n)!}{n!}.$$

Therefore,

$$\int_0^\pi \sin^{2n} \theta \, d\theta = \frac{\pi}{2^{2n}} \frac{\phi^{(n)}(0)}{n!} = \frac{\pi}{2^{2n}} \frac{1}{n!} \left[\frac{(2n)!}{n!} \right] = \frac{(2n)!}{2^{2n} (n!)^2} \pi, \quad n = 1, 2, \dots$$